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# Optimal Control and Admissible Relaxation of Uncertain Nonlinear Elliptic Systems

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In this article we study the optimal control of uncertain systems monitored by nonlinear elliptic equations. First under convexity hypotheses, we show that the min–max problem has a solution. Then we drop the convexity assumptions and we pass to the larger relaxed system. We show that this always has a solution under very general hypotheses on the data. We also produce conditions that guarantee that the relaxation is admissible; i.e., the relaxed problem is the "closure" of the original one. 

## 1. INTRODUCTION

In many engineering applications, the system under consideration is only partially known, in the sense that its dynamical equation may contain parameters or coefficients whose probability law is not known precisely, except perhaps its support. Such systems are known in the literature as "uncertain systems." The purpose of this article is to study the optimal control of nonlinear elliptic uncertain systems.

So let $Z \subseteq \mathbb{R}^N$ be a bounded domain with boundary $\partial Z = \Gamma$. By a "multi-index" $\alpha = (\alpha_1, \dots, \alpha_N)$ we understand an array of $N$ nonnegative integers. We set $|\alpha| = \sum_{k=1}^N \alpha_k$ (the length of the multi-index), $D_k = \partial/\partial z_k$ and $D^\alpha = D_1^{\alpha_1} \dots D_N^{\alpha_N}$. For $\alpha = 0$, we set $D^0 u = u$. The system under consideration is described by the following dynamical equation:

$$\sum_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha A_\alpha(z, \eta(x)(z)) + \int_W f(z, x(z), v, u(z))\,\mu(dv) = g(z)\text{a.e.} \tag{1}$$

$$D^\beta x|_\Gamma = 0, |\beta| \le m-1, u(z) \in U(z)\text{a.e.}, \qquad \mu \in M_+^1(W).$$



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In the above equation $\eta(x) = \{D^{\alpha}x : |\alpha| \leq m\}$ (i.e., $\eta(x)(\cdot)$ denotes the array of all partial derivatives up to order $m$ including $x$). The boundary condition says that all partial derivatives, up to order $m - 1$, should vanish on $\Gamma$. Precise hypotheses on the data $A_{\alpha}$, $f$, $g$, $U$ are given in Section 2. The unknown parameter takes values on a compact metric space $W$, but the underlying probability law $\mu(\cdot)$ is not known exactly. So we can only say that $\mu(\cdot) \in M_{+}^{1}(W)$, the space of probability measures on $W$. A control function $u : Z \to \mathbb{R}$ is said to be "admissible" if $u(\cdot)$ is measurable and $u(z) \in U(z)$a.e. (i.e., $u(\cdot)$ is a measurable selector of the multifunction $U(\cdot)$). Let $x(u, \mu)(\cdot)$ be a solution of (1), generated by the control function $u(\cdot)$ and the parameter distribution $\mu$. To any "state-control" pair $[x, u]$, corresponds an integral cost

$$J(x(u,\mu),u) = \int_{Z} L(z,\theta(x(u,\mu))(z),u(z))\,dz,$$

where $\theta(x) = \{D^{\beta}x : |\beta| \leq m - 1\}$. The controller adopts a pessimistic approach in that he tries to minimize the maximum possible cost. So our optimal control problem is

$$\inf\left[\sup\left(J(x(u,\mu),u) : \mu \in M_{+}^{1}(W)\right) : u \in S_{U}\right] = m, \tag{2}$$

with $S_U$ being the set of admissible controls; i.e., $S_U$ is the set of all measurable selectors of $U(\cdot)$.

Our goal is to find a triple $(\hat{x}, \hat{u}, \hat{\mu})$s.t.

$$J(\hat{x}(\hat{u},\hat{\mu}),\hat{u}) = m.$$

Such a triple is an "optimal solution" for problem (2).

In Section 2 (Theorem 1), under convexity hypotheses on the cost of integrand $L$ and the control constraint set $U$, and with the control entering linearly in the dynamics, we show that a solution of problem (2) exists. In Section 3, we turn our attention to the fully nonlinear case, with no convexities present. It is well known that in this case, even for finite-dimensional systems, an optimal solution need not exist. This leads us to the introduction of a larger system, with convexified dynamics, constraints, and cost functional, which is known as the "relaxed problem." We show that the relaxed problem always has a solution, under very general hypotheses on the data. A natural question then arises: Is it possible to approximate the relaxed optimal state with arbitrary degree of accuracy, using states of the original system? Such an approximation result guarantees that under reasonable hypotheses on the cost integrand the relaxed and original problems have the same values. If this happens, then we call the relaxation of the system "admissible." For such a relaxed system, we know that an $\epsilon$-optimal control can be found among the original (physically realizable) controls. For finite-dimensional systems with no uncertainty present, the issue of admissible relaxability was studied by Clarke [4], while for infinite-dimensional systems, it was examined by Ahmed [1] and Papageorgiou [9]. Furthermore, Papageorgiou [10] established an equivalence between relaxability and well posedness for nonlinear parabolic optimal control problems. In the past, Schmitendorf [14], Barnish [2], and Tanimoto [14] examined finite-dimensional uncertain systems (or, "problems of guaranteed performance" as they call them), but did not address the question of relaxation.

Let $K$ be a compact metric space and denote by $C(K)$ the space of continuous functions on $K$ and by $M(K)$ the space of all bounded Borel measures on $K$. From the Riesz representation theorem, we know that $C(K)^* = M(K)$. Furthermore from the Dinculeanu–Foias theorem (see for example [15]), we know that $L^1(Z, C(K))^* = L^\infty(Z, M(K))$. It is in $L^\infty(Z, M(K))$ that the relaxed controls live (see Section 3).

If $Y$ is a complete, separable metric space, a multifunction

$$U : Z \to 2^Y \setminus \{\emptyset\}$$

is measurable if for all $v \in Y$, $z \to d_Y(v, U(z)) = \inf\{d_Y(v, w) : w \in U(z)\}$ is measurable for all $v \in Y$ (here $d_Y(\cdot, \cdot)$ denotes the metric on $Y$).

Suppose $X$ is a reflexive Banach space and $A : X \to X^*$ an operator:

(i) $A$ has "property $M$" if $x_n \xrightarrow{w} x$ in $X$, $A(x_n) \xrightarrow{w} b$ in $X^*$, and $\overline{\lim}\langle A(x_n), x_n - x\rangle \le 0$ implies $A(x) = b$ (here $\langle \cdot , \cdot \rangle$ denotes the duality brackets for the pair $(X, X^*)$),

(ii) $A$ is "pseudomonotone" if $x_n \xrightarrow{w} x$ in $X$ and

$$\overline{\lim}\langle A(x_n), x_n - x\rangle \le 0$$

implies $\langle A(x), x - y\rangle < \underline{\lim}\langle A(x_n), x_n - y\rangle$ for all $y \in X$,

(iii) $A$ is "monotone" if $\langle A(x) - A(y), x - y\rangle \ge 0$ for all $x, y \in X$,

(iv) $A$ is "strongly monotone" if $\langle A(x) - A(y), x - y\rangle \ge c\,\|x - y\|^2$ for all $x$, $y \in X$, and with $c > 0$.

We know that strongly monotone $\Rightarrow$ monotone $\Rightarrow$ pseudomonotone $\Rightarrow$ property $M$. For details we refer to [7].

## 2. EXISTENCE OF OPTIMAL SOLUTION

In this section, for a particular version of system (1), in which the control appears linearly in the dynamics, we prove the existence of an optimal solution for problem (2). So the system under consideration is now the following:

$$\sum_{|\alpha|\le m} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(z, \eta(x)(z)) + \int_W f(z, x(z), v)\,\mu(dv)\,u(z) = g(z) \text{a.e.} \tag{3}$$

$$D^{\beta} x\,|_{\Gamma} = 0, |\beta| \le m-1, u(z) \in U(z) \text{a.e.} \qquad u \in M_+^1(W).$$

Again, the optimal control problem is the inf–sup problem (2).

We need the following hypotheses on the data. In what follows $N_m = (N+m)!/N!m!$ (the number of derivatives of order $\le m$),

$$N_{m-1} = \frac{(N+m-1)!}{N!(m-1)!}$$

(the number of derivatives of order $\le m-1$), and $\overline{N} = N_m - N_{m-1}$ (the number of derivatives of order $= m$).

H($A$): $A_{\alpha}: Z \times \mathbb{R}^{N_{m-1}} \times \mathbb{R}^{\overline{N}} \to \mathbb{R}$ are functions s.t.

(1) $z \to A_{\alpha}(z, \theta, \xi)$ is measurable,

(2) $(\theta, \xi) \to A_{\alpha}(z, \theta, \xi)$ is continuous,

(3) $|A_{\alpha}(z, \theta, \xi)| \le a_1(z) + c_1(\|\theta\|^{p-1} + \|\xi\|^{p-1})$ a.e. with $1 < p < \infty$, $a_1(\cdot) \in L^q(Z)$, $1/p + 1/q = 1$, and $c_1 > 0$,

(4) $\sum_{|\alpha|=m} (A_{\alpha}(z, \theta, \xi) - A_{\alpha}(z, \theta, \xi'))(\xi_{\alpha} - \xi'_{\alpha}) > 0$,

(5) $\sum_{|\alpha|\le m} A_{\alpha}(z, \theta, \xi)\xi_{\alpha} \ge c_2\|\xi\|^p - a_2(z)$ a.e. with $a_2(\cdot) \in L^1(Z)$, $c_2 > 0$.

H($f$): $f: Z \times \mathbb{R} \times W \to \mathbb{R}$ is a function s.t.

(1) $z \to f(z, x, v)$ is measurable,

(2) $|f(z, x, v) - f(z, x', v)| \le k(z)|x - x'|$ a.e. for all $v \in W$, with $k(\cdot) \in L^1(Z)$,

(3) $v \to f(z, x, v)$ is continuous,

(4) $-\beta \le f(z, x, v)ux$ for all $(z, x) \in Z \times \mathbb{R}$ and all $|u| \le M$ (sign condition),

(5) $|f(z, x, v)u| \le a_3(z) + c_3\|x\|^{p-1}$ a.e. for all $v \in W$ and all $|u| \le M$, with $a_3(\cdot) \in L^q(Z)$ and $c_3 > 0$.

H($U$): $U(z) = \{u \in \mathbb{R}: |u| \le \gamma(z)\}$ with $\gamma(\cdot)$ measurable $\gamma(z) \le M$ a.e.

H($L$): $L : Z \times \mathbb{R}^{N_{m-1}} \times \mathbb{R} \to \mathbb{R}$ is an integrand s.t.
(1) $(z, \theta, u) \to L(z, \theta, u)$ is measurable,
(2) $(\theta, u) \to L(z, \theta, u)$ is l.s.c.,
(3) $\theta \to L(z, \theta, u)$ is continuous,
(4) $u \to L(z, \theta, u)$ is concave,
(5) $|L(z, \theta, u)| \le \varphi(z) + M_1 \|\theta\|^p$ a.e. for all $|u| \le M$, with $\varphi(\cdot) \in L^1(Z)$, $M_1 > 0$.

We have the following existence result for problem (2).

THEOREM 1. *If hypotheses* H($A$), H($f$), H($U$), H($L$) *hold and* $g \in L^1(Z)$, *then problem* (2) *admits an optimal solution.*

*Proof.* Fix $u \in S_U$ (an admissible control) and then consider the maximization problem

$$\sup\left[ J(x(u, \mu), u) : \mu \in M_+^1(W) \right] = m(u). \tag{4}$$

First we solve this maximization problem. To this end, let $A : W_0^{m,p}(Z) \to W^{-m,q}(Z)$ be the nonlinear operator defined by

$$\langle A(x), y \rangle = \int_Z \sum_{|\alpha| \le m} A_\alpha(z, \eta(x)(z)) D^\alpha y(z)\, dz, \; y \in W_0^{m,p}(Z)$$

with $\langle \cdot, \cdot \rangle$ denoting the duality brackets for the pair

$$(W_0^{m,p}(Z), W^{-m,q}(Z)).$$

From Theorem 1 of [3], we know that $A(\cdot)$ is a pseudomonotone map. Also because of Hypothesis H($A$) (3), we have

$$\|A(x)\|_* \le \hat{c}_1(1 + \|x\|) \qquad \text{for some } \hat{c}_1 > 0 \text{ (boundedness)}$$

and because of Hypothesis H($A$) (5), we have

$$\langle A(x), x \rangle \ge \hat{c}_2 \|x\|^p - \hat{a} \qquad \text{for some } \hat{c}_2, \hat{a} > 0 \text{ (coercivity)}$$

(here as well as in the sequel, $\|\cdot\|$ denotes the $W_0^{m,p}(Z)$-norm, while $\|\cdot\|_*$ denotes the $W^{-m,q}(Z)$-norm). Also let $\hat{f} : L^p(Z) \times M_+^1(W) \to L^q(Z)$ be defined by

$$\hat{f}(x, \mu)(z) = \int_W f(z, x(z), v)\, \mu(dv).$$

Let $\{[x_n, \mu_n]\}_{n\geq 1}$ be a maximizing sequence for problem (4). Recall that $M_+^1(W)$ is the space of all probability measures on $W$, and when we equip it with the relative $w^*(M(W), C(W))$-topology (also known among probabilists as the weak or narrow topology), it becomes a compact metric space (see for example [11, theorem 6.4, p. 45]). So by passing to a subsequence if necessary, we may assume that $\mu_n \xrightarrow{w^*} \mu \in M_+^1(W)$. Also for every $n \geq 1$ we have

$$A(x_n) + \hat{f}(x_n, \mu_n)u = g$$

(the equality understood in $W^{-m,q}(Z)$; recall $L^q(Z) \subseteq W^{-m,q}(Z)$). Then we have

$$\langle A(x_n), x_n\rangle + \langle \hat{f}(x_n, \mu_n)u, x_n\rangle = \langle g, x_n\rangle.$$

Note that since $\hat{f}(x_n, \mu_n)u \in L^q(Z) \subset W^{-m,q}(Z)$ and $x_n \in W_0^{m,p}(Z)$, we have

$$\begin{aligned}\langle \hat{f}(x_n, \mu_n)u, x_n\rangle &= \left(\hat{f}(x_n, \mu_n)u, x_n\right)_{L^q, L^p}\\ &= \int_Z \int_W f(z, x_n(z), v)\mu(dv)u(z)x_n(z)\,dz\\ \Rightarrow -\beta|Z| &\leq \langle \hat{f}(x_n, \mu_n)u, x_n\rangle\end{aligned}$$

(see Hypothesis H($f$)($\underline{e}$)), with $|Z|$ being the Lebesgue measure of $Z \subseteq \mathbb{R}^N$.
Recall that

$$\langle A(x_n), x_n\rangle \geq \hat{c}_2\|x_n\|^p - \hat{a}.$$

So finally we have

$$\begin{aligned}&\hat{c}_2\|x_n\|^p - \hat{a} - \beta|Z| \leq \hat{c}_3\|g\|_q \cdot \|x_n\| \qquad \text{for some } \hat{c}_3 > 0\\ &\Rightarrow \hat{c}_2\|x_n\|^{p-1} - \frac{\hat{a} + \beta|Z|}{\|x_n\|} \leq \hat{c}_3\|g\|_q\end{aligned}$$

from which we deduce that $\{x_n\}_{n\geq 1}$ is bounded in $W_0^{m,p}(Z)$. So by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $W_0^{m,p}(Z)$.

Also since $\|A(x_n)\|_* \le \hat{c}_1(1 + \|x_n\|)$, $\{A(x_n)\}_{n\ge 1}$ is bounded in $W^{-m,q}(Z)$ and so we may assume that $A(x_n) \xrightarrow{w} b$ in $W^{-m,q}(Z)$. We have

$$\begin{aligned}
&\left|\int_W f(z,x_n(z),v)\mu_n(dv) - \int_W f(z,x(z),v)\mu(dv)\right| \\
&\quad \le \int_W |f(z,x_n(z),v) - f(z,x(z),v)|\,\mu_n(dv) \\
&\qquad + \left|\int_W f(z,x(z),v)\mu_n(dv) - \int_W f(z,x(z),v)\mu(dv)\right| \\
&\quad \le k(z)|x_n(z) - x(z)| + \left|\int_W f(z,x(z),v)\mu_n(dv)\right. \\
&\qquad \left. - \int_W f(z,x(z),v)\mu(dv)\right|.
\end{aligned}$$

Since $x_n \xrightarrow{w} x$ in $W_0^{m,p}(Z)$ and $W_0^{m,p}(Z)$ embeds compactly in $L^p(Z)$, we may assume (at least for a subsequence of $x_n$), that $|x_n(z) - x(z)| \to 0$ a.e. Also since $f(z,x,\cdot)$ is continuous and $\mu_n \xrightarrow{w} \mu$ in $M_+^1(W)$, we have

$$\left|\int_W f(z,x(z),v)\mu_n(dv) - \int_W f(z,x(z),v)\mu(dv)\right| \to 0 \qquad \text{as } n \to \infty.$$

Therefore $|\int_W f(z,x_n(z),v)\mu_n(dv) - \int_W f(z,x(z),v)\mu(dv)| \to 0$ a.e. on $Z$ and so by the dominated convergence theorem, we get

$$\hat{f}(x_n,\mu_n)u \xrightarrow{s} \hat{f}(x,\mu)u \qquad \text{in } L^q(Z).$$

Using this fact on the equation

$$\begin{aligned}
&\langle A(x_n), x_n - x\rangle + \langle \hat{f}(x_n,\mu_n)u, x_n - x\rangle = \langle g, x_n - x\rangle \\
&\quad \Rightarrow \lim\langle A(x_n), x_n - x\rangle = 0.
\end{aligned}$$

But recall that $A(\cdot)$ being pseudomonotone has property $M$. So $A(x) = b$; $A(x_n) \xrightarrow{w} A(x)$ in $W^{-m,q}(Z)$. Therefore in the limit as $n \to \infty$, we get

$$A(x) + \hat{f}(x,\mu)u = g \qquad (\text{in } W^{-m,q}(Z)).$$

Next recall that $W_0^{m,p}(Z)$ embeds compactly in $W^{m-1,p}(Z)$. So we may assume that $\theta(x_n)(z) \to \theta(x)(z)$ a.e. on $Z$. Then because of hypothesis H($L$) and Lebesgue's dominated convergence theorem, we get

$$m(u) = \lim \int_Z L(z,\theta(x_n)(z),u(z))dz = \int_Z L(z,\theta(x)(z),u(z))dz.$$

Next let $m = \inf[m(u) : u \in S_U]$ and take $\{u_n\}_{n \geq 1} \subseteq S_U$ to be a minimizing sequence of this problem. Then from the first part of this proof we know that every $n \geq 1$, we can find $[x_n, \mu_n] \in W_0^{m,p}(Z) \times M_+^1(W)$ s.t. $x_n = x_n(u_n, \mu_n)$ and $J(x_n(u_n, \mu_n), u_n) = m(u_n)$. As before exploiting the coercivity of $A(\cdot)$, we can deduce that $\{x_n\}_{n \geq 1}$ is bounded in $W_0^{m,p}(Z)$. So by passing to a subsequence if necessary we may assume that

$$x_n \xrightarrow{w} x \qquad \text{in } W_0^{m,p}(Z)$$

$$x_n(z) \longrightarrow x(z) \text{ a.e. on } Z \qquad \text{(compact embedding of } W_0^{m,p}(Z) \text{ in } L^p(Z))$$

$$A(x_n) \xrightarrow{w} b \text{ in } W^{-m,q}(Z) \qquad \text{(boundedness of } A)$$

$$u_n \xrightarrow{w^*} u \text{ in } L^\infty(Z) \qquad \text{(Hypothesis H}(U))$$

and

$$\mu_n \xrightarrow{w} \mu \text{ in } M_+^1(W) \qquad \left(\text{since } M_+^1(W) \text{ with the } w^*\text{-topology a compact metric space}\right).$$

Recall that $\int_W f(z, x_n(z), v)\mu_n(dv) \to \int_W f(z, x(z), v)\mu(dv)$ a.e. and by the dominated convergence theorem also in $L^q(Z)$. So

$$\langle \hat{f}(x_n, \mu_n)u_n, x_n - x\rangle \to 0 \qquad \text{as } n \to \infty.$$

Hence $\lim\langle A(x_n), x_n - x\rangle = 0 \Rightarrow A(x) = b$ (by property $M$). Thus in the limit as $n \to \infty$, we get

$$A(x) + \hat{f}(x, \mu)u = g, \qquad u \in S_U.$$

Also because of Hypothesis H($L$) and Theorem 1 of [8], we get

$$\int_Z L(z, \theta(x)(z), u(z))\,dz \leq \underline{\lim} \int_Z L(z, \theta(x_n)(z))\,dz = \underline{\lim}\, m(u_n) = m$$

$$\Rightarrow m = \int_Z L(z, \theta(x)(z), u(z))\,dz = J(x(u, \mu), u). \qquad \text{Q.E.D.}$$

## 3. RELAXED PROBLEM

If we drop the convexity hypotheses on $L(z, \theta, \cdot)$ and $U(z)$ and in the dynamics the control does not appear linearly, then in general we cannot guarantee the existence of an optimal solution for problem (2). To get a solution we need to introduce a larger, convexified version of problem (2), known as the "relaxed problem." There is no unique approach to relaxation. Here we adopt the Gamkrelidze–Warga approach, which uses transition probabilities (stochastic kernels) as relaxed controls. We show that our relaxed problem is the closure of the original one, in the sense that its trajectories are the limit points of the set of trajectories of the original system, while its cost functional captures the asymptotic behavior of the minimizing sequences of the original optimization problem.

Let $U: Z \to 2^{\mathbb{R}} \setminus \{\varnothing\}$ be a measurable multifunction with closed values in $K = [-M, M]$. Let $\Sigma: Z \to 2^{M_+^1(K)} \setminus \{\varnothing\}$ be defined by

$$\Sigma(z) = \{\lambda \in M_+^1(K) : \lambda(U(z)) = 1\}.$$

Clearly $\Sigma(\cdot)$ has nonempty, convex, and concept values in $M_+^1(W)$ (the latter equipped with the relative $w^*(M(K), C(K))$. Furthermore from Theorem 6 of [12], we get that $\Sigma(\cdot)$ is measurable. Let $S_\Sigma$ denote the set of measurable selections of $\Sigma(\cdot)$. Note that $S_U \subseteq S_\Sigma$. Just associate to each $u \in S_U$ the Dirac stochastic kernel $\delta_u$ corresponding to it (i.e., $\delta_{u(z)}(C) = \chi_C(u(z))$). Since in our formulation $S_\Sigma$ is the set of relaxed controls, we see that the original controls are a subset of the relaxed ones.

The dynamics of the relaxed system are the following:

$$\sum_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha A_\alpha(z, \eta(x)(z))$$
$$+ \int_K \int_W f(z, x(z), v, r)\, \mu(dv)\, \lambda(z)(dr) = g(z) \text{a.e.} \tag{5}$$
$$D^\beta x\,|_\Gamma = 0, \qquad |\beta| \le m-1, \qquad \lambda \in S_\Sigma, \qquad \mu \in M_+^1(W).$$

The cost functional is now

$$J_r(x(\lambda, \mu), \lambda) = \int_Z \int_K L(z, \theta(x)(z), r)\, \lambda(z)(dr)\, dz.$$

Then the relaxed optimal control problem is

$$\inf\left[\sup\left(J_r(x(\lambda, \mu), \lambda) : \mu \in M_+^1(W)\right) : \lambda \in S_\Sigma\right] = m_r. \tag{6}$$

First we show that (6) admits an optimal solution under very mild hypotheses on the data. Namely we make the following hypothesis:

H$(f)_1$: $f: Z \times \mathbb{R} \times W \times \mathbb{R} \to \mathbb{R}$ is a function s.t.

(1) $z \to f(z, x, v, r)$ is measurable,

(2) $|f(z, x, v, r) - f(z, x', v, r)| \le k(z)|x - x'|$ a.e. for all $(v, r) \in W \times K$ with $k \in L_+^\infty$,

(3) $v \to f(z, x, v, r)$ is continuous uniformly for $r \in K$,

(4) $r \to f(z, x, v, r)$ is continuous,
(5) $f(z, x, v, r) \geq -\beta$ for all $(x, v, r) \in \mathbb{R} \times W \times K$ with $\beta \geq 0$,
(6) $|f(z, x, v, r)| \leq a_3(z) + c_3|x|^{p-1}$ a.e. with $a_3(\cdot) \in L^q_+(Z)$, $c_3 \geq 0$.

H$(U)_1$: $U: Z \to 2^{\mathbb{R}} \setminus \{\varnothing\}$ is a measurable multifunction with nonempty, closed values contained in $K = [-M, M]$.

H$(L)_1$: $L: Z \times \mathbb{R}^{N_{m-1}} \times \mathbb{R} \to \mathbb{R}$ is an integrand s.t.
(1) $(z, \theta, u) \to L(z, \theta, u)$ is measurable,
(2) $(\theta, u) \to L(z, \theta, u)$ is l.s.c.,
(3) $|L(z, \theta, u)| \leq \varphi(z) + M_1|\theta|^p$ a.e. with $\varphi(\cdot) \in L^1(Z)_+$, $M_1 > 0$.

We have the following existence theorem concerning problem (6):

THEOREM 2. *If hypotheses* H$(A)$, H$(f)_1$, H$(U)_1$, H$(L)_1$ *hold and* $g \in L^q(Z)$, *then problem* (6) *admits an optimal solution.*

*Proof.* Fix an admissible control $\lambda \in S_\Sigma$. Then using a method similar to that employed in the first part of the proof of Theorem 1, we can establish the existence of $[x(\lambda, \mu), \mu] \in W_0^{m,p}(Z) \times M_+^1(W)$ s.t.

$$m_r(\lambda) = \sup\left[J_r(y(\lambda, \mu'), \lambda) : \mu' \in M_+^1(W)\right] = J_r(x(\lambda, \mu), \lambda).$$

Then consider the minimization problem

$$\inf\left[m_r(\lambda) : \lambda \in S_\Sigma\right] = m_r.$$

Let $\{\lambda_n\}_{n \geq 1} \subseteq S_\Sigma$ be a minimizing sequence for this last problem. Since $\{\lambda_n\}_{n \geq 1}$ is bounded in $L^\infty(Z, M(K))$ and since the weak* topology on bounded sets in $L^\infty(Z, M(K))$ is metrizable (because the predual $L^1(Z, C(K))$ is separable; see [5, Theorem 1, p. 426], applying Alaoglu's theorem and by passing to a subsequence if necessary, we may assume that $\lambda_n \xrightarrow{w^*} \lambda$ in $L^\infty(Z, M(K))$. But from Corollary 4, p. 377 of [13], $S_\Sigma$ is $w^*$-compact in $L^\infty(Z, M(K))$. So $\lambda \in S_\Sigma$. Let $[x_n, \mu_n] \in W_0^{m,p}(Z) \times M_+^1(W)$ s.t. $x_n = x(\lambda_n, \mu_n)$ and $m_r(\lambda_n) = J_r(x_n, \lambda_n)$, $n \geq 1$. We know (see the proof of Theorem 1) that by passing to a subsequence if necessary, we may assume that

$$x_n \xrightarrow{w} x \qquad \text{in } W_0^{m,p}(Z)$$
$$x_n(z) \longrightarrow x(z) \qquad \text{a.e. in } \mathbb{R}$$

and

$$\mu_n \xrightarrow{w^*} \mu \qquad \text{in } M_+^1(W).$$

Then for any $p(\cdot) \in L^p(Z)$, we have

$$\sup_{r\in K} \left|\left( \int_W f(z, x_n(z), v, r)\mu_n(dv) - \int_W f(z, x(z), v, r)\mu(dv)\right) p(z)\right|$$

$$\leq \sup_{r\in K} \left|\int_W (f(z, x_n(z), v, r) - f(z, x(z), vr)) p(z)\mu_n(dv)\right|$$

$$+ \sup_{r\in K} \left|\int_W f(z, x(z), v, r) p(z)(\mu_n(dv) - \mu(dv))\right|$$

$$\leq k(z)|x_n(z) - x(z)|p(z) + \sup_{r\in K} \left|\int_W f(z, x(z), v, r) p(z)\mu_n(dv)\right.$$

$$\left.- \int_W f(z, x(z), v, r) p(z)\mu(dv)\right|.$$

Let $\eta_n(z, r) = \int_W f(z, x(z), v, r)p(z)\mu_n(dv)$ and

$$\eta(z, r) = \int_W f(z, x(z), v, r) p(z)\mu(dv).$$

Clearly $\eta_n$, $\eta$ are both Caratheodory functions (i.e., measurable in $z$, continuous in $r$). Furthermore, let $r_n \in K$ s.t. $r_n \to r$ in $K$ and

$$\sup_{r\in K} |\eta_n(z, r) - \eta(z, r)| = |\eta_n(z, r_n) - \eta(z, r_n)|.$$

Then from Theorem 6.8, p. 51 of [11], and Hypothesis $H(f)_1$ (3), we have that

$$\begin{aligned} &|\eta_n(z, r_n) - \eta(z, r_n)| \to 0 && \text{a.e. on } Z \\ &\Rightarrow \eta_n(z, \cdot) \to \eta(z, \cdot) && \text{a.e. in } C(K) \end{aligned}$$

and by the dominated convergence theorem, we finally have that

$$\eta_n \xrightarrow{s} \eta \qquad \text{in } L^1(Z, C(K)).$$

Therefore if we denote by $((\cdot, \cdot))_0$ the duality brackets for the pair $(L^1(Z, C(K)), L^\infty(Z, M(K)))$, we get that

$$((\eta_n, \lambda_n))_0 \to ((\eta, \lambda))_0$$

$$\Rightarrow \int_Z \left( \int_K \int_W f(z, x_n(z), v, r)\mu_n(dv)\lambda_n(z)(dr)\right.$$

$$\left. - \int_K \int_W f(z, x(z), v, r)\mu(dv)\lambda(z)(dr)\right) p(z)\,dz \to 0$$

$$\Rightarrow \int_K \int_W f(z, x_n(z), v, r)\mu_n(dv)\lambda_n(z)(dr)$$

$$\xrightarrow{w} \int_K \int_W f(z, x(z), v, r)\mu(dv)\lambda(z)(dr) \qquad \text{in } L^q(Z).$$

As in the proof of Theorem 1, using the fact that $A$ has property $M$, we get $A(x_n) \xrightarrow{w} A(x)$ in $W^{-m,q}(Z)$. So in the limit as $n \to \infty$, we get

$$\sum_{|\alpha|\le m} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(z, \eta(x)(z)) + \int_K \int_W f(z, x(z), v, r)\,\mu(dv)\,\lambda(z)(dr) = g(z) \text{ a.e.}$$

$$D^{\beta} x\,|_{\Gamma} = 0, \qquad |\beta| \le m-1, \qquad \lambda \in S_{\Sigma}, \qquad \mu \in M_+^1(W).$$

In addition, by approximating the normal integrand $L(z, \theta, u)$, using Caratheodory integrands, as in the proof of Theorem 3.3 of [9] (see also [6]), we get

$$\begin{aligned} m_r &= \underline{\lim} \int_Z \int_K L(z, \theta(x_n)(z), r)\,\lambda_n(z)(dr)\,dz \\ &\le \int_Z \int_K L(z, \theta(x)(z), r)\,\lambda(z)(dr)\,dz = J_r(x(\lambda, \mu), \lambda) \\ &\Rightarrow J_r(x(\lambda, \mu), \lambda) = m_r. \end{aligned}$$

Q.E.D.

*Remark.* A useful by-product of the above proof is that the set $S_r$ of relaxed states is $w$-compact in $W_0^{m,p}(Z)$ (hence compact in $W_0^{m-1,p}(Z)$).

For all practical purposes, the relaxation will be admissible if every relaxed state can be approximated arbitrarily close, by states of the original system, and the values of the two optimal control problems are in fact equal.

In this last part of the article, we provide reasonable hypotheses on the data that guarantee the admissibility of the relaxation.

Our first task is to obtain an approximation result of the relaxed states, as mentioned above. We need the following addition hypothesis:

$H_0$: For every pair $[\mu, u] \in M_+^1(W) \times S_U$, we assume that system (1) has a unique state $x \in W_0^{m,p}(Z)$.

*Remark.* If we strengthen hypothesis H($A$) (4) to

$$\sum_{|\alpha|\le m} (A_{\alpha}(z, \eta) - A_{\alpha}(z, \eta'))(\eta_{\alpha} - \eta'_{\alpha}) \ge 0 \tag{4$'$}$$

$$\text{for all } \eta, \eta' \in \mathbb{R}^{N_m}$$

and in addition to H($f$)$_1$ assume that $x \to f(z, x, v, r)$ is monotone increasing, then clearly $H_0$ is satisfied. In this case $A : W_0^{m,p}(Z) \to W^{-m,q}(Z)$ is monotone.

Alternatively, if hypothesis H($A$) (4) becomes

$$(4)'' \quad \sum_{|\alpha|\le m} (A_\alpha(z,\eta) - A_\alpha(z,\eta'))(\eta_\alpha - \eta'_\alpha) \ge c_0\|\eta - \eta'\|^p_{\mathbb{R}^{N_m}}$$

$$\text{for all } \eta, \eta' \in \mathbb{R}^{N_m} \text{ with } c_0 > 0,$$

and in Hypothesis H$(f)_1$ (2), $k \in L^\infty(Z)$, with $\|k\|_\infty < c_0$, then again we can check that Hypothesis $\mathrm{H}_0$ is verified. In this case $A: W_0^{m,p}(Z) \to W^{-m,q}(Z)$ is strongly monotone. If for example, the differential operator of our problem is for $p \ge 2$, $-\sum_{k=1}^{\mathrm{N}} D_k(|D_k x|^{p-2} D_k x)$ (the pseudo-Laplacian), then via Tartar's inequality we can check that it generates a strongly monotone operator $A(\cdot)$.

Let $S$, $S_r \subseteq W_0^{m,p}(Z) \subseteq L^p(Z)$, be the sets of states of (1) and (5) respectively.

THEOREM 3. *If hypotheses* H($A$), H$(f)_1$, H$(U)_1$, $\mathrm{H}_0$ *hold and* $g \in L^q(Z)$, *then* $\overline{S} = S_r$, *the closure taken in the* $W_0^{m-1,p}(Z)$*-norm.*

*Proof.* Let $x \in S_r$. Then by definition there exists $[\mu, \lambda] \in M_+^1(W) \times S_\Sigma$ s.t.

$$A(x) + \hat{f}(x, \lambda, \mu) = g \qquad (\text{equality in } W^{-m,q}(Z))$$

with $\hat{f}(x, \lambda, \mu)(z) = \int_K \int_W f(z, x(z), v, r)\mu(v)\lambda(z)(dr)$. ▮

Using Corollary 4 of [13] (see also Theorem IV-2-6 of [15]), we know that we can find $u_n \in S_U$ s.t. $\delta_{u_n} \xrightarrow{w^*} \lambda$ in $L^\infty(Z, M(K))$. Let $x_n \in W_0^{m,p}(Z)$ be the unique state generated by $[\mu, \delta_{u_n}]$. We know that $\{x_n\}_{n\ge1}$ is bounded in $W_0^{m,p}(Z)$ and so we may assume $x_n \xrightarrow{w} y$ in $W_0^{m,p}(Z)$. As before $\hat{f}(x_n, \delta_{u_n}, \mu) \xrightarrow{w} \hat{f}(x, \lambda, \mu)$ in $L^q(Z)$ and so in the limit, we get $A(y) + \hat{f}(y, \mu, \lambda) = g$ with $[\mu, \lambda] \in M_+^1(W) \times S_\Sigma$ as above. Because of Hypothesis $\mathrm{H}_0$, $x = y$. Recalling that $W_0^{m,p}(Z)$ embeds compactly in $W_0^{m-1,p}(Z)$ (see the remark following Theorem 2), we get the desired density theorem. Q.E.D.

To establish the equality of the values of the two optimal control problems (original and relaxed), we need the following stronger hypothesis on the cost integrand $L(z, x, u)$.

H$(L)_2$: $L: Z \times \mathbb{R}^{N_{m-1}} \times \mathbb{R} \to \mathbb{R}$ is an integrand s.t.

(1) $z \to L(z, \theta, u)$ is measurable,
(2) $(\theta, u) \to L(z, \theta, u)$ is continuous,
(3) $|L(z, \theta, u)| \le \varphi(z) + M_1|\theta|^p$ a.e. for all $u \in K = [-M, M\}$, with $\varphi(\cdot) \in L^1(Z)$, $M_1 > 0$.

THEOREM 4. *If hypotheses* H($A$), H($f$)$_1$, H($U$)$_1$, H$_0$, H($L$)$_2$ *hold and* $g \in L^q(Z)$, *then* $m = m_r$.

*Proof.* From Theorem 2, we know that we can find $[x, \mu, \lambda] \in W_0^{m,p}(Z) \times M_+^1(W) \times S_\Sigma$ s.t. $x = x(\mu, \lambda)$ and $m_r = J_r(x, \mu, \lambda)$. From the proof of Theorem 3, we know that we can find $[x_n, u_n] \in W_0^{m,p}(Z) \times S_U$ s.t. $x_n = x(\mu, u_n)$ and

$$x_n \xrightarrow{w} x \qquad \text{in } W_0^{m,p}(Z)$$

and

$$\delta_{u_n} \xrightarrow{w^*} \lambda \qquad \text{in } L^\infty(Z, M(K)).$$

Then we have

$$\sup_{r \in K} |L(z, \theta(x_n)(z), r) - L(z, \theta(x)(z), r)|$$
$$= |L(z, \theta(x_n)(z), r_n) - L(z, \theta(x)(z), r_n)|$$

Let $r_n \to r$ in $K$ and also since $W_0^{m,p}(Z)$ embeds compactly in $W_0^{m-1,p}(Z)$, we may assume that $\theta(x_n) \xrightarrow{s} \theta(x)$ in $L^p(Z)^{N_{m-1}}$ and $\theta(x_n)(z) \to \theta(x)(z)$ a.e. in $\mathbb{R}^{N_{m-1}}$. So because of Hypothesis H($L$)$_2$ (2), we have

$$|L(z, \theta(x_n)(z), r_n) - L(z, \theta(x)(z), r)| \to 0 \qquad \text{as } n \to \infty$$
$$\Rightarrow \hat{L}_n(z, \cdot) \to \hat{L}(z, \cdot) \qquad \text{in } C(K),$$

where $\hat{L}_n(z, \cdot) = L(z, \theta(x_n)(z), \cdot)$. Then by the dominated convergence theorem, we get

$$\hat{L}_n \xrightarrow{s} L \qquad \text{in } L^1(Z, C(K))$$
$$\Rightarrow \left(\left(\hat{L}_n, \delta_{u_n}\right)\right)_0 \to \left((\hat{L}, \lambda)\right)_0 = m_r$$
$$\Rightarrow m \le m_r \qquad \left(\text{since } m \le \left(\left(\hat{L}_n, \delta_{u_n}\right)\right)_0, n \ge 1\right).$$

But clearly, we always have $m_r \le m \Rightarrow m = m_r$. Q.E.D

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